Bending of Circular Sandwich Plates due to Asymmetric Temperature Distribution

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I. Introduction

THERMOELASTIC behavior of sandwich plates has been considered by many investigators. However, most of these investigations pertain to the analysis of rectangular plates. Thermal deflection of circular sandwich plates was investigated by Bruun[1] and Huang and Ebcioglu,[2] but with the restriction of rotational symmetry. Nonrotationally symmetrical bending of circular sandwich plates was considered by the author[3]; however, no thermal effect was included in the analysis.

In this paper the thermoelastic equations for the bending of circular sandwich plates subjected to arbitrary temperature distribution are derived by means of the principle of stationary complementary energy. To illustrate the use of these equations an example, a clamped circular sandwich plate subjected to arbitrary temperature distribution which is expressible in the form of a double infinite series, is presented.

The assumptions made in the presented analysis are as follows. 1) Two faces are isotropic but may be different in material, thickness, and temperature. The Poisson's ratios of the faces are the same. Bending rigidities of the individual faces may be neglected. 2) The core is considered to be isotropic and is assumed to take only the transverse shear stress. 3) Small deflection theory is considered.

II. Analysis

Consider a circular sandwich plate of radius a consisting of a core of thickness h and two faces of thickness, t_1 and t_2 (Fig. 1). The origin of the coordinate axes is located at the center of the plate. The stress couples and stress resultants may be defined in terms of stresses as follows:

$$M_r = \sigma_{r2}t_2h_2 - \sigma_{r1}t_1h_1 \tag{1a}$$

$$M_\theta = \sigma_{\theta 2}t_2h_2 - \sigma_{\theta 1}t_1h_1 \tag{1b}$$

$$M_{r\theta} = -\sigma_{r\theta 2}t_2h_2 + \sigma_{r\theta 1}t_1h_1 \tag{1c}$$

$$N_r = \sigma_{r2}t_2 + \sigma_{r1}t_1 \tag{1d}$$

$$N_\theta = \sigma_{\theta 2}t_2 + \sigma_{\theta 1}t_1 \tag{1e}$$

$$N_{r\theta} = \sigma_{r\theta 2}t_2 + \sigma_{r\theta 1}t_1 \tag{1f}$$

$$Q_r = \sigma_{rz}h, Q_\theta = \sigma_{\theta z}h \tag{1g}$$

where h_i is the distance from the approximate neutral plane to the center of the ith face (Fig. 1) and is defined by[4]

$$h_1 = m[h + (1/2)(t_1 + t_2)]/(1 + m) \tag{2a}$$

$$h_2 = [h + (1/2)(t_1 + t_2)]/(1 + m) \tag{2b}$$

$$m = E_2t_2/E_1t_1 \tag{2c}$$

Figure 1 shows the positive directions of the stress resultants and stress couples.

The equilibrium equations in this case are given by

$$\partial N_r/\partial r + (1/r)\partial N_{r\theta}/\partial\theta + (N_r - N_\theta)/r = 0 \tag{3}$$

$$\partial N_{r\theta}/\partial r + (1/r)(\partial N_\theta/\partial\theta) + 2(N_{r\theta}/r) = 0 \tag{4}$$

$$\partial Q_r/\partial r + (1/r)\partial Q_\theta/\partial\theta + Q_r/r = 0 \tag{5}$$

$$\partial M_r/\partial r - (1/r)(\partial M_{r\theta}/\partial\theta) + (M_r - M_\theta)/r - Q_r = 0 \tag{6}$$

$$\partial M_{r\theta}/\partial r - (1/r)(\partial M_\theta/\partial\theta) + 2M_{r\theta}/r + Q_\theta = 0 \tag{7}$$

If $\bar{M}_r$, $\bar{M}_{r\theta}$, $\bar{Q}_r$, $\bar{N}_r$, and $\bar{N}_{r\theta}$ are the prescribed stress couples and stress resultants at the boundary then the boundary conditions at $r = a$ are

$$\bar{M}_r = M_r, \bar{M}_{r\theta} = M_{r\theta}, \bar{Q}_r = Q_r, \bar{N}_r = N_r, \bar{N}_{r\theta} = N_{r\theta} \tag{8}$$

The complementary energy of the plate, in this case, may be written as

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^a \left\{\sum_{i=1}^{2}\left[\frac{t_i}{E_i}(\sigma_{ri}^2 - 2\nu\sigma_{ri}\sigma_{\theta i} + \sigma_{\theta i}^2) + \frac{2(1+\nu)t_i}{E_i}\sigma_{r\theta i} + \alpha_i T_i(\sigma_{ri} + \sigma_{\theta i})\right] + \frac{h}{G}[\sigma_{rz}^2 + \sigma_{\theta z}^2]\right\} r dr d\theta \tag{9}$$

where

$$T_i = \int_{-t_i/2}^{t_i/2} T dz \tag{10}$$

Solving for stresses from Eqs. (1a–g) and substituting them into Eq. (9) yields

$$\begin{aligned} U = \frac{1}{2}\int_0^{2\pi}\int_0^a \Big\{ & C_1[M_r^2 - 2\nu M_rM_\theta + M_\theta^2] + C_2[2M_rN_r - 2\nu(M_rN_\theta + M_\theta N_r) + 2M_\theta N_\theta] + \\ & C_3[N_r^2 - 2\nu N_rN_\theta + N_\theta^2] + 2(1+\nu)C_1M^2_{r\theta} - 4(1+\nu)C_2M_{r\theta}N_{r\theta} + 2(1+\nu)C_3N_{r\theta}^2 + \\ & \alpha_1T_1\left[-\frac{M_r + M_\theta}{t_1(h_1+h_2)} + \frac{N_r + N_\theta}{t_1(1+m)}\right] + \alpha_2T_2\left[\frac{M_r + M_\theta}{t_2(h_1+h_2)} + \frac{m(N_r + N_\theta)}{t_2(1+m)}\right] + \frac{1}{Gh}[Q_r^2 + Q_\theta^2]\Big\} r dr d\theta \end{aligned} \tag{11}$$

where

$$C_1 = \frac{1}{E_1t_1c^2} + \frac{1}{E_2t_2c^2}$$

$$C_2 = -\frac{1}{E_1t_1c(1+m)} + \frac{m}{E_2t_2c(1+m)} \tag{12a}$$

$$C_3 = \frac{1}{E_1t_1(1+m)^2} + \frac{m^2}{E_2t_2(1+m)^2}, c = h_1 + h_2 \tag{12b}$$

Introducing Lagrangian multipliers u, v, w, α, β, $\bar{\alpha}$, $\bar{\beta}$, $\bar{u}$, $\bar{v}$, and $\bar{w}$, the auxiliary functional can be formulated as follows:

$$\begin{aligned} I = U + \int_0^{2\pi}\int_0^a \Big\{ & u\left[\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{N_{r\theta}}{\partial\theta} + \frac{N_r - N_\theta}{r}\right] + v\left[\frac{\partial N_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial N_\theta}{\partial\theta} + 2\frac{N_{r\theta}}{r}\right] + w\left[\frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial\theta} + \frac{Q_r}{r}\right] + \\ & \alpha\left[\frac{\partial M_r}{\partial r} - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} + \frac{M_r - M_\theta}{r} - Q_r\right] + \beta\left[\frac{\partial M_{r\theta}}{\partial r} - \frac{1}{r}\frac{\partial M_\theta}{\partial\theta} + 2\frac{M_{r\theta}}{r} + Q_\theta\right]\Big\} r dr d\theta + \\ & \int_0^{2\pi}[\bar{\alpha}(\bar{M}_r - M_r) + \bar{\beta}(\bar{M}_{r\theta} - M_{r\theta}) + \bar{u}(\bar{N}_r - N_r) + \bar{v}(\bar{N}_{r\theta} - N_{r\theta}) + \bar{w}(\bar{Q}_r - Q_r)]_{r=a}\, a d\theta \end{aligned} \tag{13}$$

Carrying out the first variation of I, i.e., $\delta I = 0$, and integrating by parts, the following Euler equations and the

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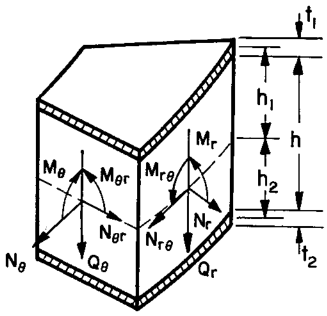

Fig. 1 Element of plate.

natural boundary conditions are obtained:

$$C_1(M_r - \nu M_\theta) + C_2(N_r - \nu N_\theta) - \frac{1}{r}\frac{\partial(\alpha r)}{\partial r} + \frac{\alpha}{r} + P = 0 \tag{14}$$

$$C_1(M_\theta - \nu M_r) + C_2(N_\theta - \nu N_r) - \frac{\alpha}{r} + \frac{1}{r}\frac{\partial \beta}{\partial \theta} + P = 0 \tag{15}$$

$$2(1+\nu)C_1 M_r - 2(1+\nu)C_2 N_{r\theta} + \frac{1}{r}\frac{\partial \alpha}{\partial \theta} - \frac{1}{r}\frac{\partial(\beta r)}{r} + \frac{2}{r}\beta = 0 \tag{16}$$

$$C_2(M_r - \nu M_\theta) + C_3(N_r - \nu N_\theta) - \frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{u}{r} + Q = 0 \tag{17}$$

$$C_2(M_\theta - \nu M_r) + C_3(N_\theta - \nu N_r) - \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{u}{r} + Q = 0 \tag{18}$$

$$2(1+\nu)C_3 N_{r\theta} - 2(1+\nu)C_2 M_{r\theta} - \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial(vr)}{\partial r} + 2\frac{v}{r} = 0 \tag{19}$$

$$(1/Gh)\, Q_r - \alpha + w/r - (1/r)[\partial(wr)]/\partial r = 0 \tag{20}$$

$$(1/Gh)\, Q_\theta + \beta - (1/r)(\partial w/\partial \theta) = 0 \tag{21}$$

$$\alpha = \bar{\alpha},\ \beta = \bar{\beta},\ u = \bar{u},\ v = \bar{v},\ w = \bar{w},\ \text{at } r = a \tag{22}$$

where

$$P = \frac{\alpha_2 T_2}{t_2 c} - \frac{\alpha_1 T_1}{t_1 c},\ Q = \frac{1}{1+m}\left[\frac{\alpha_1 T_1}{t_1} + m\frac{\alpha_2 T_2}{t_2}\right] \tag{23}$$

It can be readily seen that the Lagrange multipliers define the displacement components of sandwich plate: α and β are the rotations of the faces in the r and θ directions, respectively, u and v are the displacements at the "neutral plane" in r and θ directions, respectively, and w is the transverse displacement of the sandwich plate.

Solving for the stress couples and the stress resultants from Eqs. (14–21) yields

$$M_r = K_1\left\{\frac{\partial \alpha}{\partial r} + \nu\left(\frac{\alpha}{r} - \frac{1}{r}\frac{\partial \beta}{\partial \theta}\right) - (1+\nu)P - \frac{C_2}{C_3}[\epsilon_r + \nu\epsilon_\theta - (1+\nu)Q]\right\} \tag{24}$$

$$M_\theta = K_1\left\{\nu\frac{\partial \alpha}{\partial r} + \frac{\alpha}{r} - \frac{1}{r}\frac{\partial \beta}{\partial \theta} - (1+\nu)P - \frac{C_2}{C_3}[\nu\epsilon_r + \epsilon_\theta - (1+\nu)Q]\right\} \tag{25}$$

$$M_{r\theta} = \frac{(1-\nu)K_1}{2}\left\{\frac{\partial \beta}{\partial r} - \frac{\beta}{r} - \frac{1}{r}\frac{\partial \alpha}{\partial \theta} + \frac{C_2}{C_3}\epsilon_{r\theta}\right\} \tag{26}$$

$$N_r = K_2\left\{\frac{\partial \alpha}{\partial r} + \nu\left(\frac{\alpha}{r} - \frac{1}{r}\frac{\partial \beta}{\partial \theta}\right) - (1+\nu)P - \frac{C_1}{C_2}[\epsilon_r + \nu\epsilon_\theta - (1+\nu)Q]\right\} \tag{27}$$

$$N_\theta = K_2\left\{\nu\frac{\partial \alpha}{\partial r} + \frac{\alpha}{r} - \frac{1}{r}\frac{\partial \beta}{\partial \theta} - (1+\nu)P - \frac{C_1}{C_2}[\nu\epsilon_r + \epsilon_\theta - (1+\nu)Q]\right\} \tag{28}$$

$$N_{r\theta} = -\frac{(1-\nu)K_2}{2}\left\{\frac{\partial \beta}{\partial r} - \frac{\beta}{r} - \frac{1}{r}\frac{\partial \alpha}{\partial \theta} + \frac{C_1}{C_2}\epsilon_{r\theta}\right\} \tag{29}$$

$$Q_r = Gh\{\alpha + \partial w/\partial r\},\ Q_\theta = Gh\{-\beta + (1/r)(\partial w/\partial \theta)\} \tag{30}$$

where

$$K_1 = 1/[(C_1 - C_2^2/C_3^2)(1-\nu^2)] \tag{31}$$

$$K_2 = 1/[(C_2 - C_1 C_3/C_2)(1-\nu^2)] \tag{32}$$

$$\epsilon_r = \partial u/\partial r \tag{33}$$

$$\epsilon_\theta = (1/r)(\partial v/\partial \theta) + u/r \tag{34}$$

$$\epsilon_{r\theta} = (1/r)\partial u/\partial \theta + \partial v/\partial r - v/r \tag{35}$$

Equations (3) and (4) are satisfied identically by introducing the Airy Stress Function $F(r,\theta)$ such that

$$N_r = (1/r)\partial F/\partial r + (1/r^2)\partial^2 F/\partial \theta^2 \tag{36}$$

$$N_\theta = \partial^2 F/\partial r^2 \tag{37}$$

$$N_{r\theta} = -(1/r)(\partial^2 F/\partial r \partial \theta) + (1/r^2)(\partial F/\partial \theta) \tag{38}$$

Solving for ϵ_r, ϵ_θ, and $\epsilon_{r\theta}$ from Eqs. (27–29) with the aid of Eqs. (36–38) and substituting them into Eqs. (24–26) yields

$$M_r = D\left\{\frac{\partial \alpha}{\partial r} + \nu\left(\frac{\alpha}{r} - \frac{1}{r}\frac{\partial \beta}{\partial \theta}\right)\right\} - (1+\nu)DP - S\left\{\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right\} \tag{39}$$

$$M_\theta = D\left\{\nu\frac{\partial \alpha}{\partial r} + \frac{\alpha}{r} - \frac{1}{r}\frac{\partial \beta}{\partial \theta}\right\} - (1+\nu)DP - S\left\{\frac{\partial^2 F}{\partial r^2}\right\} \tag{40}$$

$$M_{r\theta} = \frac{1-\nu}{2}D\left\{\frac{\partial \beta}{\partial r} - \frac{\beta}{r} - \frac{1}{r}\frac{\partial \alpha}{\partial \theta}\right\} + S\left\{-\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right\} \tag{41}$$

where

$$D = 1/[C_1(1-\nu^2)],\ S = C_2/C_1 \tag{42}$$

Substituting Eq. (30) into Eq. (5) and rearranging, becomes

$$1/r\{[\partial(\alpha r)]/\partial r - \partial \beta/\partial \theta\} = -\nabla^2 w \tag{43}$$

Substituting Eqs. (30) and (39–41) into Eqs. (6) and (7), utilizing Eq. (43) and then solving for α and β yields

$$\alpha = \frac{D}{Gh}\left[-\frac{\partial}{\partial r}(\nabla^2 w) + \frac{1-\nu}{2}\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right] - \frac{(1+\nu)D}{Gh}\frac{\partial P}{\partial r} - \frac{\partial w}{\partial r} \tag{44}$$

$$\beta = \frac{D}{Gh}\left[\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 w) + \frac{1-\nu}{2}\frac{\partial \psi}{\partial r}\right] - \frac{(1+\nu)D}{Gh}\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \theta} \tag{45}$$

where ψ is a new function defined by

$$\psi = (1/r)(\partial\alpha/\partial\theta) + \partial\beta/\partial r + \beta/r \quad (46)$$

Introducing Eqs. (44) and (45) into Eq. (43) yields

$$\nabla^4 w = -(1+\nu)\nabla^2 P \quad (47)$$

which is the first governing differential equation.

The substitution of Eqs. (44) and (45) into Eq. (46) leads to the second governing differential equation

$$\nabla^2\psi - \eta^2\psi = 0 \quad (48)$$

where

$$\eta^2 = 2Gh/[D(1-\nu)] \quad (49)$$

Combining the three strains components in Eqs. (33–35) in such a way as to eliminate the displacements u and v, one obtains a compatibility equation in terms of strains. Solving for the strains from Eqs. (27–29) with the aid of Eqs. (36–38) and substituting them into the compatibility equation yields the third governing differential equation

$$\nabla^4 F = 1/(C_2^2 - C_1C_3)[\nabla^2(C_1Q - C_2P)] \quad (50)$$

The general solutions of Eqs. (47) and (48) are

$$\begin{aligned} w = w_p &+ (a_0 + b_0r^2 + C_0r^2\log r + d_0\log r) + \\ &(a_1r + b_1r^3 + C_1r^{-1} + d_1r\log r)\cos\theta + \\ &\sum_{n=2}^{\infty}(a_nr^n + b_nr^{n+2} + C_nr^{-n} + d_nr^{-n+2})\cos n\theta + \\ &(a_1'r + b_1'r^3 + C_1'r^{-1} + d_1'r\log r)\sin n\theta + \\ &\sum_{n=2}^{\infty}(a_n'r^n + b_n'r^{n+2} + C_n'r^{-n} + d_n'r^{-n+2})\sin n\theta \end{aligned} \quad (51)$$

$$\begin{aligned}\psi = e_0I_0(\eta r) + f_0K_0(\eta r) + \sum_{n=1}^{\infty}[e_nI_n(\eta r) + \\ f_nK_n(\eta r)]\cos n\theta + \sum_{n=1}^{\infty}[e_n'I_n(\eta r) + f_n'K_n(\eta r)]\sin n\theta\end{aligned} \quad (52)$$

where w_p is the particular solution whose form depends upon the form of the quantities $-(1+\nu)\nabla^2P$ and $\nabla^2[C_1Q - C_2P]/[C_2^2 - C_1C_3]$, respectively, and I_n and K_n are the modified Bessel function of first and second kinds, of order n, respectively. The homogeneous solution for F takes the same form as w except the integration constants.

III. Example

Consider a circular sandwich plate clamped along the perimeter but free to move in the plane of the plate. The boundary conditions to be applied at the support are

$$w = 0,\ \alpha = 0,\ \beta = 0,\ F = 0,\ \partial F/\partial r = 0 \text{ at } r = a \quad (53)$$

It is assumed that the temperature-related quantities $-(1+\nu)P$ and $[C_1Q - C_2P]/[C_2^2 - C_1C_3]$ are expressible in the following series form:[5]

$$-(1+\nu)P = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}P_{kn}r^k\cos n\theta + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}P_{kn}'r^k\sin n\theta \quad (54)$$

$$\frac{C_1Q - C_2P}{C_2^2 - C_1C_3} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}Q_{kn}r^k\cos n\theta + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}Q_{kn}'r^k\sin n\theta \quad (55)$$

where the coefficients P_{kn}, P_{kn}', Q_{kn}, and Q_{kn}' are known. As indicated by Forray and Newman,[5] the particular solution for w is

$$w_p = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{P_{kn}r^{k+2}\cos n\theta}{(k+2)^2 - n^2} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{P_{kn}'r^{k+2}\sin n\theta}{(k+2)^2 - n^2},\ \text{for } k+2-n \neq 0 \quad (56)$$

$$w_p = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}P_{kn}\left[\frac{\log r}{2n} - \frac{1}{(2n)^2}\right]r^n\cos n\theta + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}P_{kn}'\left[\frac{\log r}{2n} - \frac{1}{(2n)^2}\right]r^n\sin n\theta,\ \text{for } k+2-n = 0 \quad (57)$$

The particular solution for F takes the same form as w_p except P_{kn} and P_{kn}' should be replaced by Q_{kn} and Q_{kn}', respectively.

The integration constants are determined from Eqs. (53) with the aid of Eqs. (44, 45, 56, and 57), and the condition that the deflection and stress resultants remain finite at $r = 0$. Thus, for $k + n - 2 \neq 0$:

$$\begin{aligned} w = \sum_{k=0}^{\infty}\Bigg\{&\frac{P_{k0}}{(k+2)^2}\left[r^{k+2} + \frac{k}{2}a^{k+2} - \frac{k+2}{2}a^kr^2\right] + \\ &\sum_{n=1}^{\infty}\frac{1}{(k+2)^2 - n^2}\left[\left(-\frac{(k+2)-n}{\Omega_n} - 1\right)a^{k+2-n}r^n + \right. \\ &\left.\frac{(k+2)-n}{\Omega_n}a^{k-n}r^{n+2} + r^{k+2}\right]\begin{bmatrix}P_{kn}\cos n\theta \\ P_{kn}'\sin n\theta\end{bmatrix}\Bigg\} \end{aligned} \quad (58)$$

$$\psi = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{8(n+1)na^kI_n}{(k+2-n)(1-\nu)(aI_{na}')\Omega_n}\begin{bmatrix}P_{kn}'\sin n\theta \\ -P_{kn}\cos n\theta\end{bmatrix} \quad (59)$$

$$\begin{aligned}F = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(k+2)^2 - n^2}\Bigg[&\frac{k-n}{2}a^{k+2-n}r^n - \\ &\frac{k+2-n}{2}a^{k-n}r^{n+2} + r^{k+2}\Bigg]\begin{bmatrix}P_{kn}\cos n\theta \\ P_{kn}'\sin n\theta\end{bmatrix}\end{aligned} \quad (60)$$

and for $k + 2 - n = 0$:

$$w = \sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2n}\left\{\left[\log\left(\frac{r}{a}\right) - \frac{1}{\Omega_n}\right]r^n + \frac{r^{n+2}}{a^2\Omega_n}\right\}\begin{bmatrix}P_{kn}\cos n\theta \\ P_{kn}'\sin n\theta\end{bmatrix} \quad (61)$$

$$\psi = \sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{4(n+1)a^{n-2}I_n}{(1-\nu)(aI'_{na})\Omega_n}\begin{bmatrix}P_{kn}'\sin n\theta \\ -P_{kn}\cos n\theta\end{bmatrix} \quad (62)$$

$$F = \sum_{n=2}^{\infty}\sum_{k=0}^{\infty}\frac{1}{4n}\left\{\left[\log\left(\frac{r}{a}\right) - \frac{1}{\Omega_n}\right]r^n + \frac{r^{n+2}}{a^2\Omega_n}\right\}\begin{bmatrix}P_{kn}\cos n\theta \\ P_{kn}'\sin n\theta\end{bmatrix} \quad (63)$$

where

$$I_{na} = I_n(\eta r)]_{r=a},\quad I_{na}' = \left.\frac{\partial I_n}{\partial r}\right]_{r=a} \quad (64)$$

$$\Omega_n = \left(\frac{D}{Gha^2}\right)4n(n+1)n\left(\frac{I_{na}}{I'_{na}}\right) - \left(\frac{D}{Gha^2}\right)4(n+1)n - 2 \quad (65)$$

It should be noted that when G approaches infinity the quantity Ω_n equal to -2 and the expression for w becomes identical to the expression for a single layer plate without shear deformation given in Ref. 5 as it should be.

Having obtained the expressions for w, ψ, and F, the expressions for stress resultants and the stress couples may be obtained from Eqs. (36–38) and (39–41) with the aid of Eqs. (44) and (45).

References

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Heat Transfer and Pressure on a Hypersonic Blunt Cone with Mass Addition

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Nomenclature

B = dimensionless transpiration parameter, $B = \dot{m}\Delta H/q_0$
ΔH = enthalpy potential
$\dot{m}$ = mass flow of transpiration gas per unit area
p = pressure on model surface
q = heat flux to model surface
q_0 = heat flux to surface at zero transpiration
R = nose radius of model
S = surface distance along model measured from stagnation point
U_∞ = freestream velocity
ρ_∞ = freestream density

Introduction

THE thermal protection of a hypersonic vehicle is often accomplished by ablation or transpiration cooling techniques. Secondary effects, such as changes in surface pressure, can accompany the ablation or transpiration. In any case, it is known that the molecular weight of the ablated or transpired gases can strongly influence the heat transfer and pressure on the vehicle. Information, in general, both experimental and theoretical, concerning this problem is quite meager. Therefore, an experimental program was undertaken (and a theoretical program instituted) to study this problem. The theory has been described in Ref. 1. In the experimental program, the effect of molecular weight of the injected gases helium, air, argon, and freons (with molecular weight variation of 4–200) on the heat transfer and pressure on a blunt cone was studied by the transpiration technique. All gases were injected with an approximate uniform distribution along the model. A portion of the experimental work, together with some comparisons of the theories with typical data, will be described herein.

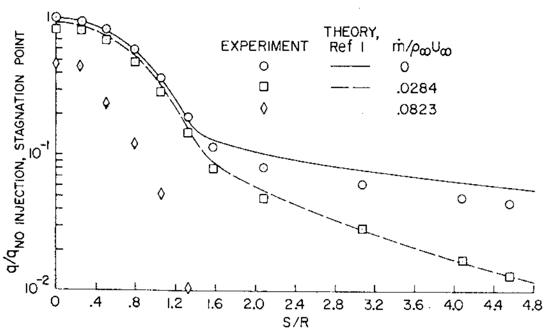

Fig. 1 Heat transfer with uniform argon injection.

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Experimental Procedures

All of the tests that are reported here were conducted in a hypersonic air stream at a nominal Mach number of 13.4, total stream enthalpy near 2100 Btu/lb, and Reynolds number of 31,000 per ft. Mass flow rate of the injected gases varied from zero to over 50% of the freestream mass flow. In all cases, the higher injected gas flow rates were sufficient to reduce stagnation point heat transfer to zero.

The test models were hemispherically blunted cones consisting of a 1-in. nose radius with a 3.5-in.-long, 7.5° semivertex angle conical afterbody. They were made of 34% porous sintered nickel with a wall thickness of 0.1 in. Thermocouples were spot welded on both inside and outside surfaces along two opposite rays of the heat-transfer model. Pressure orifices of up to 0.2 in. diam were located alternately along two rays of the pressure model. Locations of thermocouples and pressure orifices are indicated in the subsequent data presentation.

The cone wall thickness and porosity were selected to insure uniform surface mass addition over the whole model for the various external pressure distributions encountered during the tests. Local and over-all transpiration rates were measured for the models. The variation of the local mass flow rate measured over an 0.08 sq. in. area was within ±10% of the average value.

Heat transfer to the cone surface was obtained by measuring the time-temperature history of the porous wall immediately after the model was exposed to the high-speed air stream. All pertinent model temperature measurements were recorded during the 2-sec exposure time. Steady surface pressures were measured at the pressure transducers within approximately 1 sec after the model was exposed to the tunnel air stream.

Results and Discussion

An extensive amount of surface heat-transfer and pressure data on porous cones with surface mass addition of helium, air, argon, freon 22, freon 12, and freon C-318 was obtained in the experimental program. In Ref. 1, theory was used to calculate both surface heat transfer and pressure for the same test bodies but for a limited set of test conditions with argon and helium mass addition. Since an effort of this limited re-

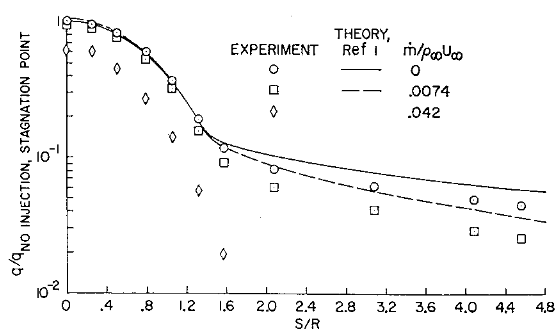

Fig. 2 Heat transfer with uniform helium injection.